

Relativistic Causal Newton Gravity Law

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Abstract. The equations of the relativistic causal Newton gravity law for the planets of the solar system are studied in the approximation when the Sun rests at the coordinates origin and the planets do not interact between each other.

1 Introduction

The Newton gravity law requires the instant propagation of the force action. The special relativity requires that the propagation speed does not exceed the speed of light. If the propagation speed is independent of the gravitating body speed, then it is equal to that of light. The special relativity requires also the gravity law covariance under Lorentz transformations. Poincaré [1] tried to find such a modification of the Newton gravity law. (Poincaré considered two mathematical problems in XX century as principal: "to create the mathematical basis for the quantum physics and for the relativity theory.") The gravity forces of two physical points should depend not on its simultaneous positions and speeds but on the positions and the speeds at the time moments which differ from each other in the time interval needed for light covering the distance between the physical points. The gravity force acting on one physical point may depend also on the acceleration of another physical point at the delayed time moment. The relativistic Newton gravity law was proposed in the paper [2]. This law for the two physical points has the form

$$\frac{d}{dt} \left(\left(1 - c^{-2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2} \frac{dx_k^\mu}{dt} \right) = -\eta^{\mu\nu} \sum_{\nu=0}^3 c^{-1} \frac{dx_k^\nu}{dt} F_{j;\mu\nu}(x_k, x_j), \quad (1.1)$$

$j, k = 1, 2, j \neq k, \mu = 0, \dots, 3$. The world line $x_k^\mu(t)$ satisfies the condition $x_k^0(t) = ct$; c is the speed of light; the diagonal 4×4 - matrix $\eta^{\mu\nu} = \eta_{\mu\nu}$, $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$; the strength $F_{j;\mu\nu}(x_k, x_j)$ is expressed through the vector potential

$$F_{j;\mu\nu}(x_k, x_j) = \frac{\partial A_{j;\nu}(x_k, x_j)}{\partial x_k^\mu} - \frac{\partial A_{j;\mu}(x_k, x_j)}{\partial x_k^\nu}, \quad (1.2)$$

$$A_{j;\mu}(x_k, x_j) = \eta_{\mu\mu} m_j G \left(\frac{d}{dt'} x_j^\mu(t') \right) \left(c |\mathbf{x}_k - \mathbf{x}_j(t')| - \sum_{i=1}^3 (x_k^i - x_j^i(t')) \frac{d}{dt'} x_j^i(t') \right)^{-1}, \quad (1.3)$$

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$$t' = c^{-1}(x_k^0 - |\mathbf{x}_k - \mathbf{x}_j(t')|), \quad j, k = 1, 2, \quad j \neq k;$$

the gravitation constant $G = (6.673 \pm 0.003) \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ and m_j is the j body mass. For a resting body world line ($x_j^0(t) = ct$ and the vector $\mathbf{x}_j(t)$ is constant) the vector potential (1.3) coincides with the Coulomb vector potential

$$A_{j;0}(x_k, x_j) = m_j G |\mathbf{x}_k - \mathbf{x}_j(c^{-1}x_k^0)|^{-1}, \quad A_{j;i}(x_k, x_j) = 0, \quad i = 1, 2, 3. \quad (1.4)$$

If the velocities of bodies are small enough to neglect their squares compared with the square of the light speed and it is possible to neglect also the time interval $c^{-1}|\mathbf{x}_k - \mathbf{x}_j(t')|$, then the vector potential (1.3) is nearly equal to the Coulomb vector potential (1.4). The vector potential (1.3) was proposed by Liénard (1898) and Wiechert (1900) as the generalization of the Coulomb vector potential (1.4). The substitution of the Coulomb vector potential (1.4) into the right-hand side of the equation (1.1) for $\mu = 1, 2, 3$ yields the right-hand side of the Newton gravity law equations. The equation (1.1) multiplied by $(1 - c^{-2}|d\mathbf{x}_k/dt|^2)^{-1/2}$ transforms as the vector. The equations (1.1), (1.2) with the Liénard - Wiechert vector potential (1.3) are the relativistic version of the Newton gravity law equations.

Sommerfeld ([3], Sec. 38): "The question may arise: what is the relativistic form of the Newton gravity law? If the law is supposed to have a vector form, this question is wrong. The gravitational field is not a vector field. It has the incomparably complicated tensor structure." The Newton gravity law equations and the equations (1.1) - (1.3) define the interactions. The body interacts only with another body. If two bodies create the common gravitational field with the vector potential $A_{1;\mu}(x, x_1) + A_{2;\mu}(x, x_2)$, any body should interact with itself and we obtain the infinity in the equations (1.3), (1.4) at $x_k = x_j$. The notion of gravitational field with the vector potential $A_{1;\mu}(x, x_1) + A_{2;\mu}(x, x_2)$ is not compatible with the Newton gravity law and with the relativistic Newton gravity law (1.1) - (1.3).

The delay $c^{-1}|\mathbf{x}_k - \mathbf{x}_j(t')|$ in the relation (1.3) provides the causality condition according to which some event in the system can influence the evolution of the system in the future only and can not influence the behavior of the system in the past, in the time preceding the given event. The delay $c^{-1}|\mathbf{x}_k - \mathbf{x}_j(t')|$ in the relation (1.3) is very important: one celestial body is a good distance off another celestial body. Poincaré [1]: "It turned out to be necessary to consider this hypothesis more attentively and to study the changes it makes in the gravity laws in particular. First, it obviously enables us to suppose that the gravity forces propagate not instantly but at the speed of light." The general relativity does not take into account the causality condition and the delay.

99.87% of the total mass of the solar system belongs to the Sun. We consider the relativistic causal Newton gravity law equations [2] for the planets of the solar system in the natural approximation when the Sun rests at the coordinates origin and the planets do not interact between each other. In this approximation the problem of planet relativistic motion was solved in the paper [2]. The planet orbits were given by the formulas which differ from the formulas defining the ellipses in the precession coefficients only. The precession coefficients for the solar system planets are practically equal to one. The similar orbits with another precession coefficients are considered in the general relativity ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)). In the beginning of the XVII century Johannes Kepler by making use of Tycho Brahe (1546 - 1601) astronomical observations found that the planet orbits are elliptic in the coordinate system where the Sun rests (Nicolaus Copernicus (1543)). The intensive astronomic observations from the middle of the XIX century and the radio-location after 1966 discovered the advances of orbit perihelion for different planets.

In the general relativity the observed value for the Mercury's perihelion advance is obtained by means of addition the advance of Mercury's perihelion ([4], Chap. 40, Sec. 40.5, Appendix 40.3) calculated in the Newton gravity theory and the advance of Mercury's perihelion calculated for the orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)). The orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) are the approximate solutions of the geodesic equation for the chosen metrics ([4], Chap. 40, Sec. 40.1, relation (40.3)). It is not obvious that we can add the advance of Mercury's perihelion obtained for the orbits ([4], Chap. 40, Sec. 40.5, Appendix 40.3) and for the orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)). It seems natural to obtain the advance of Mercury's perihelion, observed from the Earth, by making use of the Mercury and Earth orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) calculated without Newton gravity theory. In order to calculate the advance of Mercury's perihelion we need to know also the time dependence of the orbit ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) radius. In this paper we study the similar orbits of the relativistic causal Newton gravity law [2]. We shall show in this paper that the value of the Mercury's perihelion advance, observed from the Earth, depends on the perihelion angles of the Mercury and Earth orbits. The perihelion angle of the planet's orbit depends on the planet's perihelion point due to the precession coefficient in the planet's orbit formula. For the experimental verification of the equations (1.1) - (1.3) the perihelion angles of the Mercury and Earth orbits are needed.

2 Causal Coulomb and Newton laws

The relativistic Lagrange law is the particular case of the relativistic Newton second law

$$mc \frac{dt}{ds} \frac{d}{dt} \left(\frac{dt}{ds} \frac{dx^\mu}{dt} \right) + qc^{-1} \sum_{k=0}^N \sum_{\alpha_1, \dots, \alpha_k=0}^3 \eta^{\mu\mu} F_{\mu\alpha_1 \dots \alpha_k}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{dt} \dots \frac{dt}{ds} \frac{dx^{\alpha_k}}{dt} = 0, \quad (2.1)$$

$$\frac{dt}{ds} = (c^2 - |\mathbf{v}|^2)^{-1/2}, \quad v^i = \frac{dx^i}{dt}, \quad i = 1, 2, 3.$$

where $\mu = 0, \dots, 3$ and the world line $x^\mu(t)$ satisfies the condition: $x^0(t) = ct$. The force is the polynomial of the speed in the equation (2.1). It is necessary to define the series convergence for the force as an infinite series of the speed. The second relation (2.1) implies the identities

$$\sum_{\alpha=0}^3 \eta_{\alpha\alpha} \left(\frac{dt}{ds} \frac{dx^\alpha}{dt} \right)^2 = 1, \quad \sum_{\alpha=0}^3 \eta_{\alpha\alpha} \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{d}{dt} \left(\frac{dt}{ds} \frac{dx^\alpha}{dt} \right) = 0. \quad (2.2)$$

The equation (2.1) and the second identity (2.2) imply

$$\sum_{k=0}^N \sum_{\alpha_1, \dots, \alpha_{k+1}=0}^3 F_{\alpha_1 \dots \alpha_{k+1}}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{dt} \dots \frac{dt}{ds} \frac{dx^{\alpha_{k+1}}}{dt} = 0. \quad (2.3)$$

Let the functions $F_{\alpha_1 \dots \alpha_{k+1}}(x)$ satisfy the equation (2.3). Then three equations (2.1) for $\mu = 1, 2, 3$ are independent

$$m \frac{d}{dt} \left((1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - qc^{-1} \sum_{k=0}^N (c^2 - |\mathbf{v}|^2)^{-(k-1)/2} \times \left(\sum_{\alpha_1, \dots, \alpha_k=0}^3 F_{i\alpha_1 \dots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \dots \frac{dx^{\alpha_k}}{dt} \right) = 0, \quad i = 1, 2, 3. \quad (2.4)$$

The following lemma is proved in the paper [2].

Lemma. *Let there exist a Lagrange function $L(\mathbf{x}, \mathbf{v}, t)$ such that for any world line $x^\mu(t)$, $x^0(t) = ct$, the relation*

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} &= m \frac{d}{dt} \left((1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - qc^{-1} \sum_{k=0}^N \\ &\quad (c^2 - |\mathbf{v}|^2)^{-(k-1)/2} \sum_{\alpha_1, \dots, \alpha_k=0}^3 F_{i\alpha_1 \dots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \dots \frac{dx^{\alpha_k}}{dt} \end{aligned} \quad (2.5)$$

holds for any $i = 1, 2, 3$. Then the Lagrange function has the form

$$L(\mathbf{x}, \mathbf{v}, t) = -mc^2(1 - c^{-2} |\mathbf{v}|^2)^{1/2} + q \sum_{i=1}^3 A_i(\mathbf{x}, t) c^{-1} v^i + q A_0(\mathbf{x}, t) \quad (2.6)$$

and the coefficients $F_{i\alpha_1 \dots \alpha_k}(x)$ in the equations (2.4) are

$$F_{i\alpha_1 \dots \alpha_k}(x) = 0, \quad k \neq 1, \quad i = 1, 2, 3, \quad \alpha_1, \dots, \alpha_k = 0, \dots, 3, \quad (2.7)$$

$$\begin{aligned} F_{ij}(x) &= \frac{\partial A_j(\mathbf{x}, t)}{\partial x^i} - \frac{\partial A_i(\mathbf{x}, t)}{\partial x^j}, \quad i, j = 1, 2, 3, \\ F_{i0}(x) &= \frac{\partial A_0(\mathbf{x}, t)}{\partial x^i} - \frac{1}{c} \frac{\partial A_i(\mathbf{x}, t)}{\partial t}, \quad i = 1, 2, 3. \end{aligned} \quad (2.8)$$

We define the coefficients

$$F_{00}(x) = 0, \quad F_{0i}(x) = -F_{i0}(x), \quad i = 1, 2, 3. \quad (2.9)$$

Then the identity

$$\sum_{\alpha, \beta=0}^3 F_{\alpha\beta}(x) \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{dx^\beta}{dt} = 0 \quad (2.10)$$

of the type (2.3) holds. By making use of the second identity (2.2) and the relations (2.8) - (2.10) we can rewrite the equation (2.4) with the coefficients (2.7), (2.8) as the relativistic Newton second law with Lorentz force

$$\begin{aligned} mc \frac{dt}{ds} \frac{d}{dt} \left(\frac{dt}{ds} \frac{dx^\mu}{dt} \right) &= -q \eta^{\mu\mu} \sum_{\nu=0}^3 F_{\mu\nu}(x) c^{-1} \frac{dt}{ds} \frac{dx^\nu}{dt}, \\ F_{\mu\nu}(x) &= \frac{\partial A_\nu(\mathbf{x}, t)}{\partial x^\mu} - \frac{\partial A_\mu(\mathbf{x}, t)}{\partial x^\nu}, \quad \mu, \nu = 0, \dots, 3. \end{aligned} \quad (2.11)$$

For the relativistic Lagrange law the interaction is defined by the product of the charge q and the external vector potential $A_\mu(\mathbf{x}, t)$.

Let a distribution $e_0(x) \in S'(\mathbf{R}^4)$ with support in the closed upper light cone be a fundamental solution of the wave equation

$$-(\partial_x, \partial_x) e_0(x) = \delta(x), \quad (\partial_x, \partial_x) = \left(\frac{\partial}{\partial x^0} \right)^2 - \sum_{i=1}^3 \left(\frac{\partial}{\partial x^i} \right)^2. \quad (2.12)$$

We prove the uniqueness of the equation (2.12) solution in the class of distributions with supports in the closed upper light cone. Let the equation (2.12) have two solutions $e_0^{(1)}(x)$,

$e_0^{(2)}(x)$. Since its supports lie in the closed upper light cone, the convolution is defined. Now the convolution commutativity

$$\int d^4x d^4y e_0^{(2)}(x-y) e_0^{(1)}(y) \phi(x) = \int d^4x d^4y e_0^{(1)}(x) e_0^{(2)}(y) \phi(x+y) \quad (2.13)$$

implies these distributions coincidence:

$$e_0^{(j)}(x) = -(\partial_x, \partial_x) \int d^4y e_0^{(k)}(x-y) e_0^{(j)}(y), \quad (2.14)$$

$j, k = 1, 2, j \neq k$. Due to the book ([5], Sect. 30) this unique causal distribution is

$$e_0(x) = -(2\pi)^{-1} \theta(x^0) \delta((x, x)), \quad (2.15)$$

$$(x, y) = x^0 y^0 - \sum_{k=1}^3 x^k y^k, \quad \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The relativistic causal Coulomb law is given by the equations of the type (2.11)

$$m_k \frac{d}{dt} \left(\left(1 - c^{-2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2} \frac{dx_k^\mu}{dt} \right) = -q_k \eta^{\mu\mu} \sum_{\nu=0}^3 c^{-1} \frac{dx_k^\nu}{dt} F_{j;\mu\nu}(x_k, x_j), \quad (2.16)$$

$j, k = 1, 2, j \neq k$, where the strength $F_{j;\mu\nu}(x_k, x_j)$ is given by the relation (1.2) with the Liénard - Wiechert vector potential of the type (1.3)

$$\begin{aligned} A_{j;\mu}(x_k, x_j) &= -4\pi q_j K \sum_{\nu=0}^3 \eta_{\mu\nu} \int dt e_0(x_k - x_j(t)) \frac{dx_j^\nu(t)}{dt} = \\ &- q_j K \eta_{\mu\mu} \left(\frac{d}{dt} x_j^\mu(t) \right) \left(c |\mathbf{x}_k - \mathbf{x}_j(t)| - \sum_{i=1}^3 (x_k^i - x_j^i(t)) \frac{d}{dt} x_j^i(t) \right)^{-1} \Big|_{t=t(0)}, \\ &x_k^0 - ct(0) = |\mathbf{x}_k - \mathbf{x}_j(t(0))|. \end{aligned} \quad (2.17)$$

Here K is the constant of the causal electromagnetic interaction for two particles with the charges q_j . The support of the distribution (2.15) lies in the upper light cone boundary. The interaction speed is equal to that of light. It is easy to prove the second relation (2.17) by making change of the integration variable

$$x_k^0 - ct(r) = (|\mathbf{x}_k - \mathbf{x}_j(t(r))|^2 + r)^{1/2}. \quad (2.18)$$

For $r = 0$ the relation (2.18) coincides with the third relation (2.17).

The equations (2.16), (1.2), (2.17) are the relativistic causal version of the Coulomb law. The Lorentz invariant distribution (2.15) defines the delay. The Lorentz invariant solutions of the equation (2.12) are described in the paper [2]. By making use of these solutions it is possible to describe the Lorentz covariant equations of the type (2.16), (1.2), (2.17). The equations (2.16), (1.2), (2.17) are Lorentz covariant and causal due to the distribution (2.15). The quantum version of the equations (2.16), (1.2), (2.17) is defined in the paper [6]. The solutions of these causal equations do not contain the diverging integrals similar to the diverging integrals of the quantum electrodynamics.

For a world line $x_j^\mu(t)$ we define the vector proportional to $-\eta^{\mu\mu}(\partial_x, \partial_x)A_{j;\mu}(x, x_j)$

$$J^\mu(x, x_j) = -(\partial_x, \partial_x) \int dt e_0(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \int dt \delta(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \left(\frac{d}{dx^0} x_j^\mu(c^{-1}x^0) \right) \delta(\mathbf{x} - \mathbf{x}_j(c^{-1}x^0)), \mu = 0, \dots, 3. \quad (2.19)$$

The condition $x_j^0(t) = ct$ implies the continuity equation

$$\begin{aligned} \frac{\partial}{\partial x^0} J^0(x, x_j) &= - \sum_{i=1}^3 \left(\frac{d}{dx^0} x_j^i(c^{-1}x^0) \right) \frac{\partial}{\partial x^i} \delta(\mathbf{x} - \mathbf{x}_j(c^{-1}x^0)), \\ \frac{\partial}{\partial x^i} J^i(x, x_j) &= \left(\frac{d}{dx^0} x_j^i(c^{-1}x^0) \right) \frac{\partial}{\partial x^i} \delta(\mathbf{x} - \mathbf{x}_j(c^{-1}x^0)), i = 1, 2, 3, \end{aligned} \quad (2.20)$$

$$\sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} J^\mu(x, x_j) = 0. \quad (2.21)$$

The integration of the relation

$$e_0(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \int d^4y e_0(x - y) \delta(y - x_j(t)) \frac{dx_j^\mu(t)}{dt} \quad (2.22)$$

along the world line $x_j^\mu(t)$ yields

$$\int dt e_0(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \int d^4y e_0(x - y) J^\mu(y, x_j). \quad (2.23)$$

The relations (2.21), (2.23) imply the gauge condition for the vector potential (2.17)

$$\sum_{\mu=0}^3 \eta^{\mu\mu} \frac{\partial}{\partial x^\mu} A_{j;\mu}(x, x_j) = 0. \quad (2.24)$$

Due to the gauge condition (2.24) the tensor (1.2), (2.17) satisfies Maxwell equations with the current proportional to the current (2.19).

The substitution $K = -G$ and two positive or two negative gravitational masses $q_1 = \pm m_1$, $q_2 = \pm m_2$ into the equations (2.16), (1.2), (2.17) yields the relativistic causal Newton gravity law (1.1) - (1.3). By changing the constants $K = -G$, $q_1 = \pm m_1$, $q_2 = \pm m_2$ in the equations from the paper [6] we have the quantum version of the equations (1.1) - (1.3). The substitution $K = -G$ and also one positive and one negative gravitational masses $q_1 = \pm m_1$, $q_2 = \mp m_2$ into the equations (2.16), (1.2), (2.17) yields the galaxies scattering with an acceleration. Einstein [7]: "The theoretical physicists studying the problems of the general relativity can hardly doubt now that the gravitational and electromagnetic fields should have the same nature."

3 Advance of Mercury's perihelion

Due to the paper [2] the relativistic causal Newton gravity law for the solar system has the form

$$\frac{d}{dt} \left(\left(1 - c^{-2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2} \frac{dx_k^\mu}{dt} \right) = -\eta^{\mu\mu} \sum_{\nu=0}^3 c^{-1} \frac{dx_k^\nu}{dt} \sum_{j=1, \dots, 10, j \neq k} F_{j;\mu\nu}(x_k, x_j). \quad (3.1)$$

We give the number $k = 1$ for Mercury, the number $k = 2$ for Venus, the number $k = 3$ for the Earth, the number $k = 4$ for Mars, the number $k = 5$ for Jupiter, the number $k = 6$ for Saturn, the number $k = 7$ for Uranus, the number $k = 8$ for Neptune, the number $k = 9$ for Pluto and the number $k = 10$ for the Sun.

99.87% of the total mass of the solar system belongs to the Sun. We consider the Sun resting at the coordinates origin (Nicolaus Copernicus (1543)). Substituting the Sun world line $x_{10}^0(t) = ct$, $x_{10}^i(t) = 0$, $i = 1, 2, 3$, into the equalities (1.2), (1.3) we have

$$F_{10;ij}(x; x_{10}) = 0, \quad i, j = 1, 2, 3, \quad F_{10;i0}(x; x_{10}) = -m_{10}G|\mathbf{x}|^{-3}x^i, \quad i = 1, 2, 3. \quad (3.2)$$

Substituting the Sun world line $x_{10}^0(t) = ct$, $x_{10}^i(t) = 0$, $i = 1, 2, 3$, and the elliptic orbits into the expressions (1.2) and (1.3) it is possible to show that the values of strengths $F_{10;i0}(x_k; x_{10})$ considerably exceed the values of strengths $F_{j;i\nu}(x_k; x_j)$ for any $k, j = 1, \dots, 9$, $k \neq j$. It is possible to show also that the values of strengths $F_{j;i\nu}(x_{10}; x_j)$ are negligible. We neglect the action of any planet on all of the other planets and the Sun. Then the Sun rests at the coordinates origin. Due to the relations (3.2) in the coordinates system where the Sun rests at the coordinates origin the first nine equations (3.1) have the form

$$\frac{d}{dt} \left(\left(1 - c^{-2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2} \frac{dx_k^i}{dt} \right) = -m_{10}G|\mathbf{x}_k|^{-3}x_k^i, \quad i = 1, 2, 3, \quad k = 1, \dots, 9 \quad (3.3)$$

It is shown in the paper [2] that the following values

$$M_l(\mathbf{x}_k) = \sum_{i,j=1}^3 \epsilon_{ijl} \left(x_k^i \frac{dx_k^j}{dt} - x_k^j \frac{dx_k^i}{dt} \right) \left(1 - \frac{1}{c^2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2}, \quad l = 1, 2, 3, \quad (3.4)$$

$$E(\mathbf{x}_k) = c^2 \left(1 - \frac{1}{c^2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2} - m_{10}G|\mathbf{x}_k|^{-1}, \quad k = 1, \dots, 9. \quad (3.5)$$

are conserved for the equations (3.3). The antisymmetric in all indices tensor ϵ_{ijl} has the normalization $\epsilon_{123} = 1$. The conservation of the vector (3.4) is the relativistic second Kepler law. The vector \mathbf{x}_k is orthogonal to the constant vector (3.4). We introduce the polar coordinates in the plane orthogonal to the vector (3.4)

$$x_{\perp;k}^1(t) = r_k(t) \cos \phi_k(t), \quad x_{\perp;k}^2(t) = r_k(t) \sin \phi_k(t), \quad k = 1, \dots, 9. \quad (3.6)$$

Let the constants (3.4), (3.5) satisfy the inequalities

$$c^2|\mathbf{M}(\mathbf{x}_k)|^2 - m_{10}^2G^2 > 0, \quad (3.7)$$

$$|\mathbf{M}(\mathbf{x}_k)|^2((E(\mathbf{x}_k))^2 - c^4) + m_{10}^2G^2c^4 > 0, \quad k = 1, \dots, 9. \quad (3.8)$$

Due to the paper [2] the equations (3.3) have the solutions

$$\frac{p_k}{r_k(t)} = 1 + e_k \cos(\gamma_k(\phi_k(t) - \phi_{k;0})) \quad (3.9)$$

where $\phi_{k;0}$ is the constant perihelion angle and the constants

$$\begin{aligned} p_k &= (c^2|\mathbf{M}(\mathbf{x}_k)|^2 - m_{10}^2G^2)(m_{10}GE(\mathbf{x}_k))^{-1}, \\ e_k &= (c^2|\mathbf{M}(\mathbf{x}_k)|^2((E(\mathbf{x}_k))^2 - c^4) + m_{10}^2G^2c^4)^{1/2}(m_{10}GE(\mathbf{x}_k))^{-1}, \\ \gamma_k &= (c^2|\mathbf{M}(\mathbf{x}_k)|^2 - m_{10}^2G^2)^{1/2}(c|\mathbf{M}(\mathbf{x}_k)|)^{-1}, \quad k = 1, \dots, 9. \end{aligned} \quad (3.10)$$

The equations (3.9) are the relativistic first Kepler law. The orbit (3.9) is not periodic in general. The substitution of the vector (3.6) with $r_k(t) = a_k$, $\phi_k(t) = \omega_k(t - t_{k;0})$ in the equations (3.3) yields the relativistic third Kepler law:

$$(1 - c^{-2}a_k^2\omega_k^2)^{-1/2}a_k^3\omega_k^2 = m_{10}G. \quad (3.11)$$

The equations (3.9) define the trajectory of motion but do not define the time dependence of this trajectory. Let the constants (3.4), (3.5) satisfy the inequality (3.8) and the inequalities

$$(E(\mathbf{x}_k))^2 < c^4, \quad k = 1, \dots, 9. \quad (3.12)$$

Due to the paper [2] the equations (3.3) have the solutions with the constant parameter $\xi_{k;0}$

$$\begin{aligned} r_k(\xi_k) &= m_{10}GE(\mathbf{x}_k)(c^4 - (E(\mathbf{x}_k))^2)^{-1} \\ &\times \left(1 + e_k \sin((c^4 - (E(\mathbf{x}_k))^2)^{1/2}c^{-1}\xi_k)\right), \\ t_k(\xi_k) &= m_{10}G(E(\mathbf{x}_k))^2c^{-1}(c^4 - (E(\mathbf{x}_k))^2)^{-3/2} \\ &\times \left(c^3(E(\mathbf{x}_k))^{-2}(c^4 - (E(\mathbf{x}_k))^2)^{1/2}(\xi_k - \xi_{k;0})\right. \\ &\left. - e_k \cos((c^4 - (E(\mathbf{x}_k))^2)^{1/2}c^{-1}\xi_k)\right), \quad k = 1, \dots, 9. \end{aligned} \quad (3.13)$$

Let us express the constants in the equations (3.9), (3.13) through the astronomical orbit data. The orbit eccentricities: $e_1 = 0.21$, $e_2 = 0.007$, $e_3 = 0.017$, $e_4 = 0.093$, $e_5 = 0.048$, $e_6 = 0.056$, $e_7 = 0.047$, $e_8 = 0.009$, $e_9 = 0.249$. Therefore $0 < e_k < 1$, $k = 1, \dots, 9$. Let us suppose $E(\mathbf{x}_k) > 0$, $k = 1, \dots, 9$. The curve (3.9) is an ellipse with a precession. The focus of this ellipse is the coordinates origin. The major and minor "semi - axes" are equal to

$$a_k = p_k(1 - e_k^2)^{-1} = m_{10}GE(\mathbf{x}_k)(c^4 - (E(\mathbf{x}_k))^2)^{-1}, \quad (3.14)$$

$$b_k = a_k(1 - e_k^2)^{1/2} = (c^2|\mathbf{M}(\mathbf{x}_k)|^2 - m_{10}^2G^2)^{1/2}(c^4 - (E(\mathbf{x}_k))^2)^{-1/2}, \quad k = 1, \dots, 9. \quad (3.15)$$

The inequalities (3.7) and $e_k^2 < 1$ imply the inequality (3.12). Hence the Eqs. (3.13) hold. For the parameters $\xi_{k;\pm} = \pm(\pi/2)c(c^4 - (E(\mathbf{x}_k))^2)^{-1/2}$ we have the extremal radii

$$r_k(\xi_{k;\pm}) = m_{10}GE(\mathbf{x}_k)(c^4 - (E(\mathbf{x}_k))^2)^{-1}(1 \pm e_k), \quad k = 1, \dots, 9. \quad (3.16)$$

Hence, the "period" of the motion along the ellipse (3.9) is equal to

$$T_k = 2|t_k(\xi_{k;+}) - t_k(\xi_{k;-})| = 2\pi m_{10}Gc^3(c^4 - (E(\mathbf{x}_k))^2)^{-3/2}, \quad k = 1, \dots, 9. \quad (3.17)$$

Let us define the mean "angular frequency" $\omega_k = 2\pi T_k^{-1}$. The relation (3.17) implies

$$\begin{aligned} \omega_k &= (c^4 - (E(\mathbf{x}_k))^2)^{3/2}(m_{10}Gc^3)^{-1}, \\ (E(\mathbf{x}_k))^2 &= c^2(c^2 - (\omega_k m_{10}G)^{2/3}), \quad k = 1, \dots, 9. \end{aligned} \quad (3.18)$$

The substitution of the expression (3.18) into the equality (3.14) yields

$$m_{10}G = \omega_k^2 a_k^3 \left(2^{-1}(1 + \sigma_k(1 - (2a_k\omega_k c^{-1})^2)^{1/2})\right)^{-3/2}, \quad \sigma_k = \pm 1, \quad k = 1, \dots, 9. \quad (3.19)$$

Let $c \rightarrow \infty$. Then $m_{10}G = \omega_k^2 a_k^3 (2^{-1}(1 + \sigma_k))^{-3/2}$. For $\sigma_k = 1$ this expression agrees with the third Kepler law $m_{10}G = \omega_k^2 a_k^3$. Choosing $\sigma_k = 1$ in the relation (3.19) we get the "relativistic third Kepler law" for the orbit (3.9)

$$\omega_k^{-2} a_k^{-3} m_{10}G = \left(2^{-1}(1 + (1 - 4\omega_k^2 a_k^2 c^{-2})^{1/2})\right)^{-3/2} \approx 1 + \frac{3}{2}\omega_k^2 a_k^2 c^{-2}, \quad k = 1, \dots, 9. \quad (3.20)$$

According to the book ([4], Chap. 25, Sec. 25.1, Appendix 25.1) the values $\omega_k^2 a_k^3 c^{-2} = 1477m$ for $k = 1, 2, 3, 4, 6$ (Mercury, Venus, the Earth, Mars and Saturn), the values $\omega_l^2 a_l^3 c^{-2} = 1478m$ for $l = 5, 8$ (Jupiter and Neptune), the value $\omega_7^2 a_7^3 c^{-2} = 1476m$ for Uranus, the value $\omega_9^2 a_9^3 c^{-2} = 1469m$ for Pluto; the major semi-axes $a_1 = 0.5791 \cdot 10^{11}m$, $a_2 = 1.0821 \cdot 10^{11}m$, $a_3 = 1.4960 \cdot 10^{11}m$, $a_4 = 2.2794 \cdot 10^{11}m$, $a_5 = 7.783 \cdot 10^{11}m$, $a_6 = 14.27 \cdot 10^{11}m$, $a_7 = 28.69 \cdot 10^{11}m$, $a_8 = 44.98 \cdot 10^{11}m$, $a_9 = 59.00 \cdot 10^{11}m$. The values $\omega_k^2 a_k^2 c^{-2} = a_k^{-1} \cdot \omega_k^2 a_k^3 c^{-2}$, $k = 1, \dots, 9$, are negligible and therefore the Sun mass values (3.20) obtained in the relativistic Kepler problem good agrees with values $\omega_k^2 a_k^3$ obtained in Kepler problem.

The substitution of the expression (3.20) into the equality (3.18) yields

$$c^{-4}(E(\mathbf{x}_k))^2 = 1 - 2\omega_k^2 a_k^2 c^{-2} \left(1 + (1 - 4\omega_k^2 a_k^2 c^{-2})^{1/2}\right)^{-1} \approx 1 - \omega_k^2 a_k^2 c^{-2}, \quad k = 1, \dots, 9. \quad (3.21)$$

By making use of the relations (3.10), (3.14), (3.15), (3.20), (3.21) we have

$$\gamma_k = \left(1 + 4\omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1} \left(1 + (1 - 4\omega_k^2 a_k^2 c^{-2})^{1/2}\right)^{-2}\right)^{-1/2} \approx 1 - 2^{-1} \omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1}, \quad k = 1, \dots, 9. \quad (3.22)$$

The value $2^{-1} \omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1} \approx 1 - \gamma_k$ is maximal for Mercury: $1 - \gamma_1 \approx 1.3341 \cdot 10^{-8}$. The precession coefficients (3.22) of the orbits (3.9) are practically equal to one for all planets. It agrees with Tycho Brahe astronomical observations used by Kepler. For a hundred years (415 "periods" of Mercury) the advance of Mercury's perihelion is nearly $(1 - \gamma_1) \cdot 360 \cdot 415 \cdot 3600'' \approx 7''.175$. The relations (3.9), (3.22) imply the perihelion angle

$$\phi_{k;l} \approx \phi_{k;0} + 2\pi l (1 + 2^{-1} \omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1}), \quad l = 0, \pm 1, \pm 2, \dots \quad (3.23)$$

The substitution of the relations (3.14), (3.22) into the equality (3.9) yields

$$e_k \cos \left(\left(1 - 2^{-1} \omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1}\right) (\phi_k(t) - \phi_{k;0}) \right) \approx a_k (1 - e_k^2) r_k^{-1}(t) - 1, \quad (3.24)$$

$k = 1, \dots, 9$. In the general relativity the orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) are the approximate solutions of the geodesic equation for the chosen metrics ([4], Chap. 40, Sec. 40.1, relation (40.3)). These orbits are the orbits (3.24) with the perihelion angles $\phi_{k;0} = 0$ and with the precession coefficients $1 - 3\omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1}$ instead of the precession coefficients $1 - 2^{-1} \omega_k^2 a_k^2 c^{-2} (1 - e_k^2)^{-1}$. It seems that the perihelion angles are missed in [4]. We note that $3\omega_1^2 a_1^2 c^{-2} (1 - e_1^2)^{-1} \cdot 360 \cdot 415 \cdot 3600'' \approx 6(1 - \gamma_1) \cdot 360 \cdot 415 \cdot 3600'' \approx 6 \cdot 7''.175 = 43''.05$. Does the orbit (3.24), $k = 1$, or the orbit ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) agree with the observed Mercury's orbit? From the paper ([8], p. 361) we know: "Observations of Mercury are among the most difficult in positional astronomy. They have to be made in the daytime, near noon, under unfavorable conditions of the atmosphere; and they are subject to large systematic and accidental errors arising both from this cause and from the shape of the visible disk of the planet. The planet's path in Newtonian space is not an ellipse but an exceedingly complicated space-curve due to the disturbing effects of all of the other planets. The calculation of this curve is a difficult and laborious task, and significantly different results have been obtained by different computers."

Substituting the relations (3.20), (3.21) in the equality (3.13) and introducing the parameter without physical measure we get

$$\begin{aligned} a_k^{-1} r_k(\tau_k) &\approx 1 + e_k \sin \tau_k, \\ \omega_k t_k(\tau_k) &\approx \tau_k - \tau_{k;0} - e_k (1 - \omega_k^2 a_k^2 c^{-2}) \cos \tau_k, \\ \tau_k &= \omega_k a_k \xi_k \left(1 + 2^{-1} \omega_k^2 a_k^2 c^{-2}\right) \quad k = 1, \dots, 9. \end{aligned} \quad (3.25)$$

If we neglect the values $\omega_k^2 a_k^2 c^{-2}$, then the solutions (3.24), (3.25) of the equations (3.3) coincide with the solutions of the Kepler problem. Let us define the constant $\tau_{k;0}$ in the second equality (3.25) by choosing the initial time moment $t_k(0) = 0$. Then the equalities (3.25) have the form

$$\begin{aligned} a_k^{-1} r_k(\tau_k) &\approx 1 + e_k \sin \tau_k, \\ \omega_k t_k(\tau_k) &\approx \tau_k - e_k(1 - \omega_k^2 a_k^2 c^{-2})(\cos \tau_k - 1), \quad k = 1, \dots, 9. \end{aligned} \quad (3.26)$$

Let the direction of the first axis be orthogonal to the vector $\mathbf{M}(\mathbf{x}_1)$. Let the direction of the third axis coincide with the direction of vector $\mathbf{M}(\mathbf{x}_3)$. Then the second axis lies in the plane stretched on the vectors $\mathbf{M}(\mathbf{x}_1)$ and $\mathbf{M}(\mathbf{x}_3)$. Due to the relations (3.6)

$$\begin{aligned} x_1^1(t) &= r_1(t) \cos \phi_1(t), \quad x_1^2(t) = -r_1(t) \cos \theta_1 \sin \phi_1(t), \quad x_1^3(t) = r_1(t) \sin \theta_1 \sin \phi_1(t), \\ x_3^1(t) &= r_3(t) \cos \phi_3(t), \quad x_3^2(t) = r_3(t) \sin \phi_3(t), \quad x_3^3 = 0 \end{aligned} \quad (3.27)$$

where the inclination of Mercury orbit plane $\theta_1 = 7^\circ$ and the values $r_k(t), \phi_k(t)$, $k = 1, 3$, satisfy the equations (3.24), (3.26). For the definition of Mercury and the Earth trajectories it is necessary to define the perihelion angles $\phi_{1;0}, \phi_{3;0}$ in the equations (3.24).

"Observations of Mercury do not give the absolute position of the planet in space but only the direction of a line from the planet to the observer." ([8], p. 363.) The advance of Mercury's perihelion is given by the angle

$$\begin{aligned} \cos \alpha &= \frac{(\mathbf{x}_1(t_1(\tau_{1,1})) - \mathbf{x}_3(t_3(\tau_{3,1}))), \mathbf{x}_1(t_1(\tau_{1,2})) - \mathbf{x}_3(t_3(\tau_{3,2})))}{|\mathbf{x}_1(t_1(\tau_{1,1})) - \mathbf{x}_3(t_3(\tau_{3,1}))| |\mathbf{x}_1(t_1(\tau_{1,2})) - \mathbf{x}_3(t_3(\tau_{3,2}))|}, \\ c(t_3(\tau_{3,k}) - t_1(\tau_{1,k})) &= |\mathbf{x}_1(t_1(\tau_{1,k})) - \mathbf{x}_3(t_3(\tau_{3,k}))|, \quad k = 1, 2, \\ t_1(\tau_{1,2}) - t_1(\tau_{1,1}) &\leq 100T_3 \leq t_1(\tau_{1,2}) - t_1(\tau_{1,1}) + T_1 \end{aligned} \quad (3.28)$$

where the parameters $\tau_{1,1}, \tau_{1,2}$ are defined by Mercury's perihelion points, the parameters $\tau_{3,1}, \tau_{3,2}$ are the solutions of the second equation (3.28), the numbers T_1, T_3 are the orbit "periods" of Mercury and the Earth. The quotient T_3/T_1 of the Earth and Mercury orbit "periods" is approximately equal to 4.15.

By making use of the equations (3.26) we obtain the parameters corresponding to Mercury's perihelion points:

$$\begin{aligned} a_1^{-1} r_1(\tau_{1,k}) &\approx 1 - e_1, \\ \omega_1 t_1(\tau_{1,k}) &\approx \pi(2l_k + 3/2) + e_1(1 - \omega_1^2 a_1^2 c^{-2}), \\ \tau_{1,k} &\approx \pi(2l_k + 3/2), \quad k = 1, 2, \end{aligned} \quad (3.29)$$

where l_k are the integers. The first relation (3.29) coincides with the equality (3.16).

According to the book ([4], Chap. 25, Sec. 25.1, Appendix 25.1) $c^{-1}\omega_1 = 275.8 \cdot 10^{-17} m^{-1}$, $c^{-1}\omega_3 = 66.41 \cdot 10^{-17} m^{-1}$, $a_1 = 0.5791 \cdot 10^{11} m$, $a_3 = 1.4960 \cdot 10^{11} m$. The substitution of the second equality (3.29) into the third relation (3.28) yields $l_2 - l_1 = 415$.

Due to the second relation (3.28)

$$\mathbf{x}_3(t_3(\tau_{3,k})) = \mathbf{x}_3(t_1(\tau_{1,k})) + c^{-1} |\mathbf{x}_1(t_1(\tau_{1,k})) - \mathbf{x}_3(t_3(\tau_{3,k}))| \mathbf{v}_3(t'_{3,k}), \quad k = 1, 2. \quad (3.30)$$

The Earth speed is small compared with the speed of light: $c^{-1}|\mathbf{v}_3| \approx c^{-1}\omega_3 a_3 \approx 0.9935 \cdot 10^{-4}$. We neglect this value ($\arcsin 10^{-4} \approx 0^\circ.0057$). Then the relations (3.28), (3.30) imply

$$\cos \alpha \approx \frac{(\mathbf{x}_1(t_1(\tau_{1,1})) - \mathbf{x}_3(t_1(\tau_{1,1}))), \mathbf{x}_1(t_1(\tau_{1,2})) - \mathbf{x}_3(t_1(\tau_{1,2})))}{|\mathbf{x}_1(t_1(\tau_{1,1})) - \mathbf{x}_3(t_1(\tau_{1,1}))| |\mathbf{x}_1(t_1(\tau_{1,2})) - \mathbf{x}_3(t_1(\tau_{1,2}))|} \quad (3.31)$$

where the parameters $\tau_{1,k}$, $k = 1, 2$, are given by the third relation (3.29) and the relation $l_2 = l_1 + 415$.

Let us consider Mercury's perihelion points corresponding to the integers $l_1 = 0$ and $l_2 = 415$. The substitution of the values corresponding to the Mercury's perihelion, defined by the first equation (3.29), into the equation (3.24) yields

$$\begin{aligned} a_1^{-1} r_1(\pi(2l + 3/2)) &\approx 1 - e_1, \\ \phi_1(t_1(\pi(2l + 3/2))) &\approx \phi_{1;0} + 2\pi l \left(1 + 2^{-1} \omega_1^2 a_1^2 c^{-2} (1 - e_1^2)^{-1}\right) \end{aligned} \quad (3.32)$$

since the value $\omega_1^2 a_1^2 c^{-2} \approx 2.5509 \cdot 10^{-8}$ is negligible. We substitute the time, defined by the second relation (3.29), into the second relation (3.26) for the Earth

$$\begin{aligned} a_3^{-1} r_3(\tau_3(l)) &\approx 1 + e_3 \sin \tau_3(l), \\ \omega_3 t_1(\pi(2l + 3/2)) &\approx \omega_3 \omega_1^{-1} \left(\pi(2l + 3/2) + e_1(1 - \omega_1^2 a_1^2 c^{-2})\right) \approx \\ &\tau_3(l) - e_3(1 - \omega_3^2 a_3^2 c^{-2})(\cos \tau_3(l) - 1). \end{aligned} \quad (3.33)$$

Solving the second equation (3.33) we get $\tau_3(0) \approx 1.1748$, $\tau_3(415) \approx 629.09$. Substituting these values in the first equation (3.33) we have $a_3^{-1} r_3(\tau_3(0)) \approx 1.0157$, $a_3^{-1} r_3(\tau_3(415)) \approx 1.0118$. We substitute the first equation (3.33) in the equation (3.24) for the Earth

$$\cos \left(\left(1 - \frac{\omega_3^2 a_3^2 c^{-2}}{2(1 - e_3^2)} \right) (\phi_3(t_1(\pi(2l + 3/2)))) - \phi_{3;0} \right) \approx - \frac{e_3 + \sin \tau_3(l)}{1 + e_3 \sin \tau_3(l)}. \quad (3.34)$$

The function in the right - hand side of the equation (3.34) is monotonic with respect to the variable e_3 on the interval $0 \leq e_3 \leq 1$. Calculating the values of this function at the points $e_3 = 0, 1$ we get the estimation for the module of this function which implies that the equation (3.34) has a solution. Substituting the solutions $\tau_3(l)$, $l = 0, 415$, of the second equation (3.33) in the equation (3.34) we get the angles in radians

$$\begin{aligned} \phi_3(t_1(\pi(2 \cdot 0 + 3/2))) &\approx \phi_{3;0} + 2.7521, \\ \phi_3(t_1(\pi(2 \cdot 415 + 3/2))) &\approx \phi_{3;0} + 2.3544 + 2\pi \cdot 99 \left(1 + 2^{-1} \omega_3^2 a_3^2 c^{-2} (1 - e_3^2)^{-1}\right) \end{aligned} \quad (3.35)$$

since the value $\omega_3^2 a_3^2 c^{-2} \approx 0.9870 \cdot 10^{-8}$ is negligible. Substituting the radii and the angles (3.32), the radii (3.33) and the angles (3.35) in the equations (3.27), (3.31) we get the equation

$$\begin{aligned} \cos \alpha(0, 415) &\approx (((1 - e_1) \cos \phi_{1;0} - 1.0157 a_3 a_1^{-1} \cos(\phi_{3;0} + 2.7521)) \\ &\quad \times ((1 - e_1) \cos(\phi_{1;0} + 415\pi \omega_1^2 a_1^2 c^{-2} (1 - e_1^2)^{-1}) \\ &\quad - 1.0118 a_3 a_1^{-1} \cos(\phi_{3;0} + 2.3544 + 99\pi \omega_3^2 a_3^2 c^{-2} (1 - e_3^2)^{-1})) \\ &\quad + (0.99255(1 - e_1) \sin \phi_{1;0} + 1.0157 a_3 a_1^{-1} \sin(\phi_{3;0} + 2.7521)) \\ &\quad \times (0.99255(1 - e_1) \sin(\phi_{1;0} + 415\pi \omega_1^2 a_1^2 c^{-2} (1 - e_1^2)^{-1}) \\ &\quad + (1.0118 a_3 a_1^{-1} \sin(\phi_{3;0} + 2.3544 + 99\pi \omega_3^2 a_3^2 c^{-2} (1 - e_3^2)^{-1})) \\ &\quad + 0.01485(1 - e_1)^2 \sin \phi_{1;0} \sin(\phi_{1;0} + 415\pi \omega_1^2 a_1^2 c^{-2} (1 - e_1^2)^{-1}))) \\ &\quad \times (((1 - e_1) \cos \phi_{1;0} - 1.0157 a_3 a_1^{-1} \cos(\phi_{3;0} + 2.7521))^2 \\ &\quad + (0.99255(1 - e_1) \sin \phi_{1;0} + 1.0157 a_3 a_1^{-1} \sin(\phi_{3;0} + 2.7521))^2 \end{aligned}$$

$$\begin{aligned}
& +0.01485(1-e_1)^2 \sin^2 \phi_{1;0})^{-1/2} \\
& \times (((1-e_1) \cos(\phi_{1;0} + 415\pi\omega_1^2 a_1^2 c^{-2}(1-e_1^2)^{-1}) \\
& -1.0118a_3 a_1^{-1} \cos(\phi_{3;0} + 2.3544 + 99\pi\omega_3^2 a_3^2 c^{-2}(1-e_3^2)^{-1}))^2 \\
& + (0.99255(1-e_1) \sin(\phi_{1;0} + 415\pi\omega_1^2 a_1^2 c^{-2}(1-e_1^2)^{-1}) \\
& + 1.0118a_3 a_1^{-1} \sin(\phi_{3;0} + 2.3544 + 99\pi + \omega_3^2 a_3^2 c^{-2}(1-e_3^2)^{-1}))^2 \\
& + 0.01485(1-e_1)^2 \sin^2(\phi_{1;0} + 415\pi\omega_1^2 a_1^2 c^{-2}(1-e_1^2)^{-1}))^{-1/2}. \tag{3.36}
\end{aligned}$$

The perihelion angles $\phi_{1;0}$, $\phi_{3;0}$ are needed. For the orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) the perihelion angles $\phi_{k;0} = 0$. Let the perihelion angles $\phi_{1;0}$, $\phi_{3;0}$ in the equation (3.36) be equal to zero. Then $\alpha(0, 415) = 17^\circ.889$. According to the book ([4], Chap. 40, Sec. 40.5, Appendix 40.3), the observed advance of Mercury's perihelion is $1^\circ.55548 \pm 0^\circ.00011$ for a hundred years. The angle $\alpha(0, 415) = 17^\circ.889$ is not small. In our opinion for the experimental verification of the general relativity it is necessary to obtain the advance of Mercury's perihelion, observed from the Earth, by making use of the Mercury and Earth orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) calculated without Newton gravity theory. The orbits ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) are the orbits (3.24) with the perihelion angles $\phi_{k;0} = 0$ and with the precession coefficients $1 - 3\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1}$ instead of the precession coefficients $1 - 2^{-1}\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1}$. It seems that the perihelion angles are missed in [4]. In order to calculate the advance of Mercury's perihelion we need to know also the time dependence of the orbit ([4], Chap. 40, Sec. 40.5, relations (40.17), (40.18)) radius.

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